Cuckoo Hashing with a Stash: Alternative Analysis, Simple Hash Functions

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Cuckoo Hashing

Maintain a dynamic dictionary for n keys

- lookups: $\mathcal{O}(1)$
- deletions: $\mathcal{O}(1)$
- insertions: $\mathcal{O}(1)$ amortized expected
- space: $2(1+\varepsilon)n$ slots

Not so good: Insertion of a key set of size n **fails** and rebuilds the whole data structure with probability $\mathcal{O}(n^{-1})$.

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Motivation (Kirsch et al. [KMW09])

In some applications, e.g.,

- high-performance routing (packet statistics)
- database indexing

a failure probability of $\mathcal{O}(n^{-3})$ could already lead to a failure rate that is too high.

$\Rightarrow$ **Cuckoo hashing not applicable**, although its performance is suitable for such applications.

Task

Preserve the performance and lower the failure probability.

Cuckoo Hashing with a Stash

Kirsch, Mitzenmacher and Wieder [KMW09]:

- add a small constant-sized piece of memory, the so-called stash
- move elements that cannot be inserted to this stash

They prove: Using a stash of size s lowers failure probability from

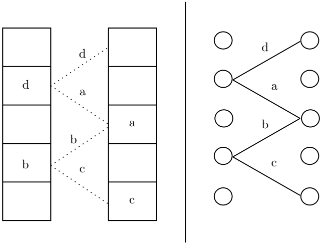
$$\mathcal{O}(n^{-1}) \text{ to } \mathcal{O}(n^{-(s+1)}).$$

Proof is technically involved ("Poissonization", "Markov Chain coupling").
Assumes fully random hash functions.

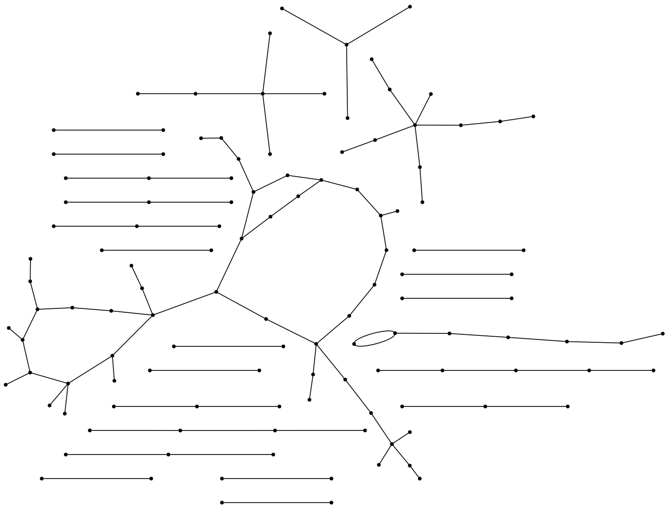
The Cuckoo Graph

The cuckoo graph $G(S, h_1, h_2)$:

- an undirected bipartite multigraph (L, R, E) where L and R represent the table cells
- $E = \{(h_1(x), h_2(x)) \mid x \in S\}$



The Cuckoo Graph - Example



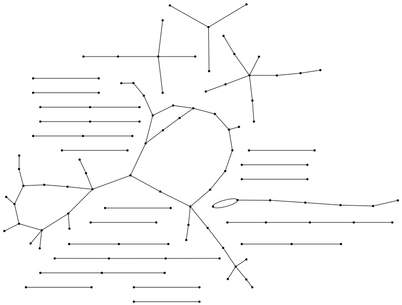
The Cuckoo Graph - Example (II)

Question: Will all key insertions be successful?

Lemma (Devroye, Morin [DM03])

The hash functions h_1 and h_2 successfully insert all keys in S if and only if each connected component of $G(S, h_1, h_2)$ is either a tree or unicyclic.

Answer: No.



How a Stash Helps

- resolves infinite loops by moving a key to the stash
- cuckoo graph contains only trees and unicyclic components if we remove stash keys

Important question

How many items are stored in the stash after a key set S of size n is inserted?

Can we find the answer for this in the cuckoo graph?

The Size of the Stash

Definition

The excess $\mathrm{ex}(G)$ is the minimal number of edges we have to remove from G such that all connected components in G contain at most one cycle.

Proposition (Kirsch et al. [KMW09])

After the insertion of S there are exactly $\mathrm{ex}(G(S, h_1, h_2))$ keys in the stash.

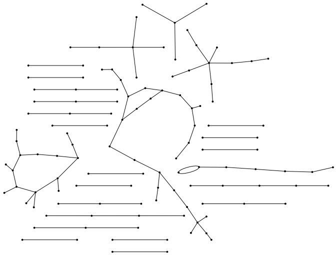
The Size of the Stash – Example

We assume stash of small constant size s.

Central Question

How likely is it that more than s keys are moved into the stash?

Equivalent question: $\Pr(\mathrm{ex}(G) > s) = \,?$



Part 1: New Proof (based on [DW03])

We know: If stash size s is not sufficient, then $\mathrm{ex}(G(S, h_1, h_2)) > s$.
Idea: Concentrate on subgraph with excess $s + 1$.

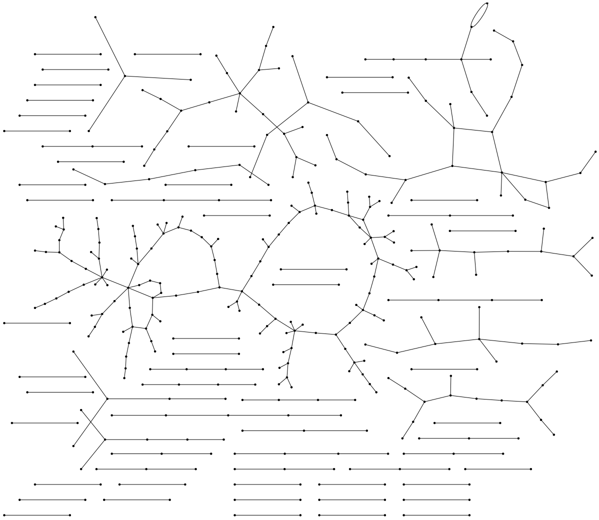
Definition

An *excess-*$(s+1)$ *core structure* of $G = G(S, h_1, h_2)$ is a subgraph G' of G with the following properties:

1. G' has excess exactly $s + 1$.
2. G' has no leaf edges.
3. G' contains only components with at least two cycles.

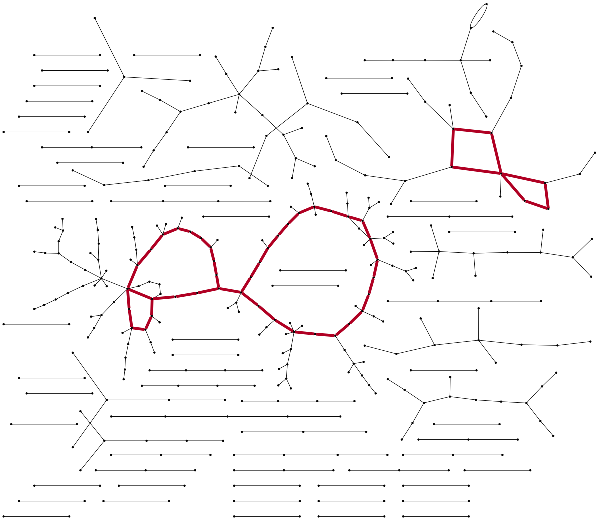
Pretty obvious: Stash of size s overflows $\Leftrightarrow$ cuckoo graph contains an excess-$(s + 1)$ core structure.

Analysis of Stash Size – Example



Question: Stash size 2 sufficient?

Analysis of Stash Size – Example



Answer: No, we can find an excess-3 core structure.

Alternative Approach to Analysis
(used in D., Woelfel [DW03])

- **count** non-isomorphic graphs that form an excess-$(s+1)$ core structure
- **bound probability** that one of the excess core structures is realized

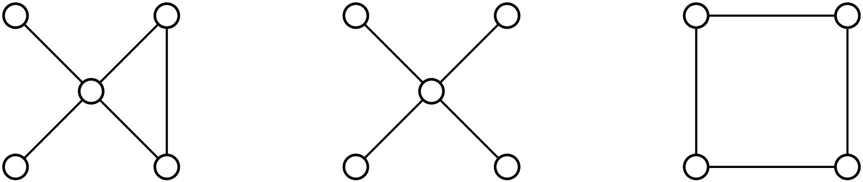
Counting Excess Structures

Theorem (Dietzfelbinger, Woelfel [DW03])

Number of non-isomorphic connected graphs with $k - \ell$ inner edges, ℓ leaf edges and cyclomatic number q is bounded by

$$k^{\mathcal{O}(\ell+q)}.$$

Problem: Excess might be shared over more than one component.
Solution: Insert edges between components to connect the graph!



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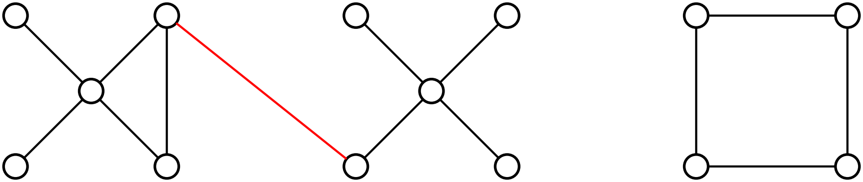

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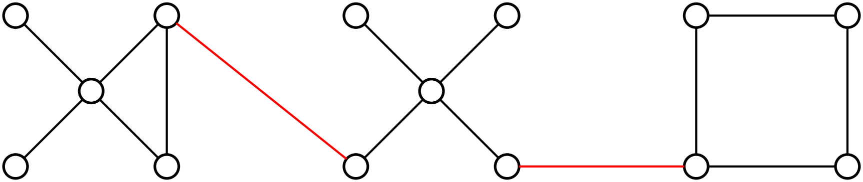

Probabilistic Analysis of the Stash Size

Result

$$\Pr(\mathrm{ex}(G) > s) = \mathcal{O}\left(n^{-(s+1)}\right).$$

The result

- matches the obtained upper bound in [KMW09]
- uses an intuitive approach to obtain the bound

Part 2: "Realistic" Hash Functions

Question

Analysis adaptable using hash functions with a bounded degree of independence, e.g., d-wise independent hash functions, which can be efficiently evaluated (like polynomials of degree $d-1$)?

Class of Hash Functions [DW03]

- $g : U \to [r]$ from d-wise independent class
- $f_1, f_2 : U \to [m]$ from d-wise independent class
- $z_0^{(1)}, \ldots, z_{r-1}^{(1)}$ and $z_0^{(2)}, \ldots, z_{r-1}^{(2)}$ random from $[m]$, tabulated

Hash functions:

$$h_1(x) = \left(f_1(x) + z_{g(x)}^{(1)}\right) \mod m$$
$$h_2(x) = \left(f_2(x) + z_{g(x)}^{(2)}\right) \mod m$$

Evaluation in constant time! Class of these hash functions: $\hat{\mathcal{R}}_{r,m}^{d}$.

Full Randomness with HF's from $\hat{\mathcal{R}}$

Theorem 2

Let $T \subseteq U$. Let $|g(T)| \geq |T| - \ell$ for $(h_1, h_2) \in \hat{\mathcal{R}}^{2\ell}_{r,m}$.
Then all $(h_1(x), h_2(x)), x \in T$, are uniformly and independently distributed in $[m]^2$.

Full Randomness on Excess Core Structures

We need full randomness on excess-$(s+1)$ core structures to reuse previous analysis.

- Define $G(S, h_1, h_2)$ to be ℓ-bad if there exists $T \subseteq S$ with $|g(T)| < |T| - \ell$ and $K(T) = G(T, h_1, h_2)$ forms an excess core structure for excess $s+1$.
- Then: If $G(S, h_1, h_2)$ is "good" then hash function pair works fully randomly on all excess core structures of our interest $\Rightarrow$ can reuse analysis!
- Question: How likely is it that $G(S, h_1, h_2)$ is good?

Bounding Probability of ℓ-bad Graphs

Problem: Pair of hash functions does not work fully randomly on bad graphs, because $|g(T)| < |T| - \ell$.

- works fully randomly on all $T' \subset T : |g(T')| \geq |T'| - \ell$
- extract subgraph $K(T')$ with $|g(T')| = |T'| - \ell$, so-called 2ℓ-reduced subgraph

Extracting 2ℓ-reduced subgraphs: Peeling

Approach: G is ℓ-bad $\Rightarrow$ graphs contains an excess core structure $K(T)$ with excess $s+1$ and $|g(T)| < |T| - \ell$.
Goal: $|g(T)| = |T| - \ell$.
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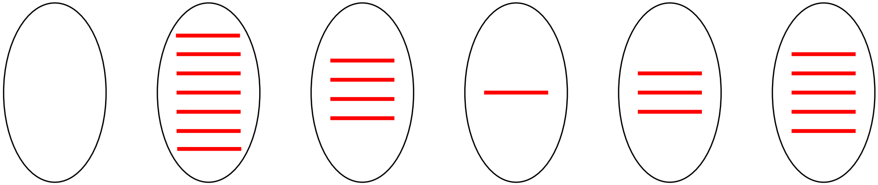

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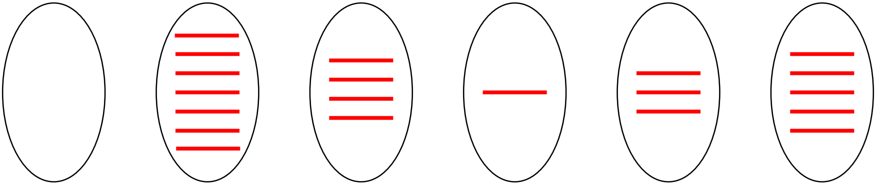

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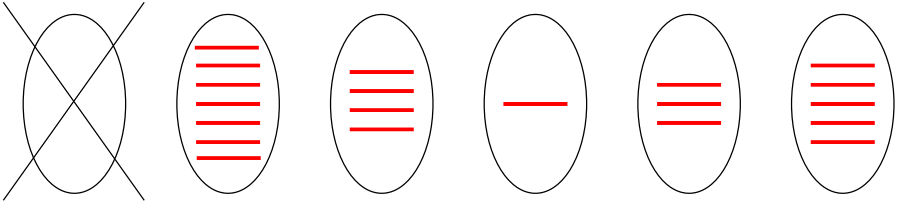

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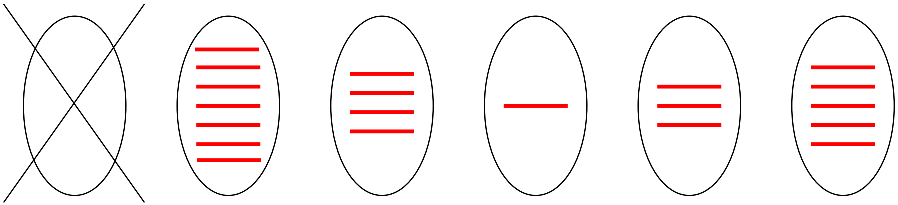
3. *Remove components while* $|g(T)| \leq |T| - \ell$.



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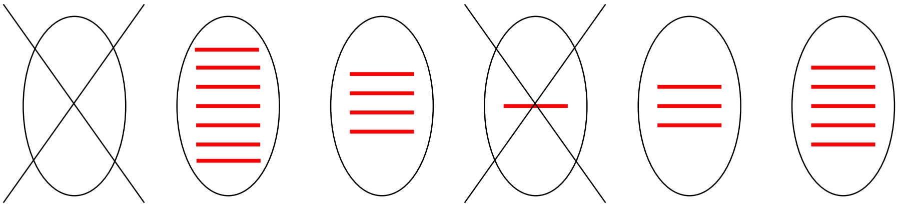

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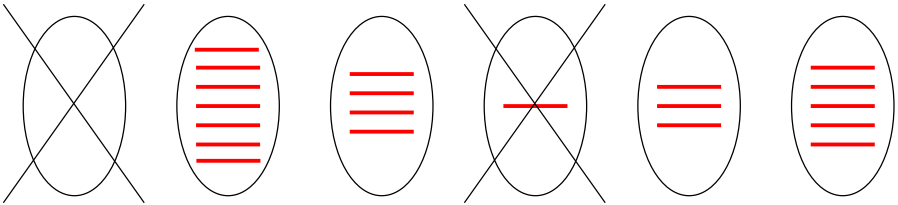

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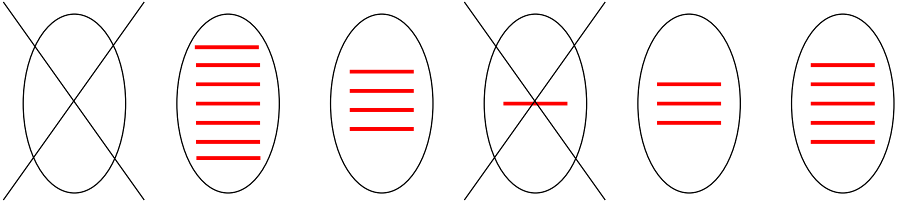

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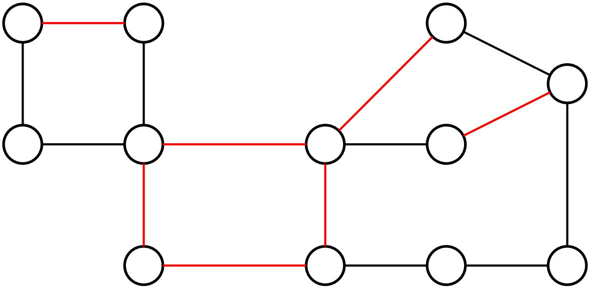

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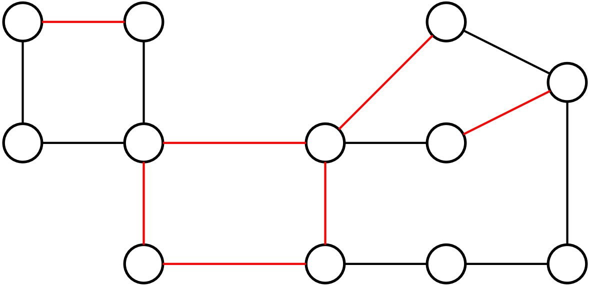

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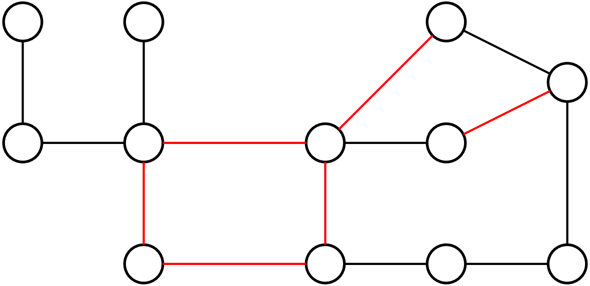

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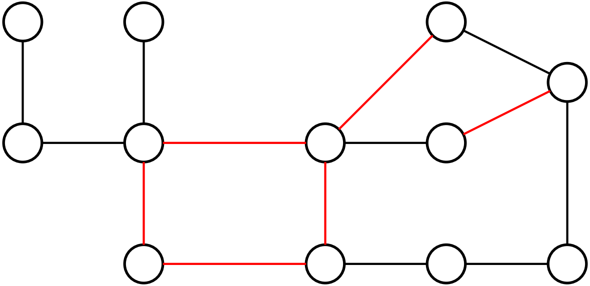

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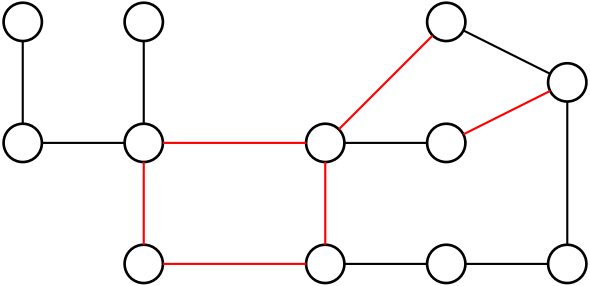
6. *Remove edges until all leaf and cycle edges are marked.*



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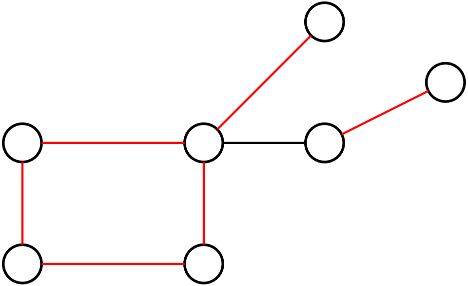


Extracting 2ℓ-reduced subgraphs – Result

Lemma 3

If G is ℓ-bad, then there exists a subset $T \subseteq S$ such that $|g(T)| = |T| - \ell$ and $K(T)$ has the following properties:

1. There is one connected component in $K(T)$ that has at most 2ℓ leaf and cycle edges.
2. All other connected components do not have leaves.
3. There are at most 2ℓ connected components.

To bound probability for ℓ-bad subgraphs: Can now re-use counting approach and have an extra factor $\mathcal{O}(r^{-\ell})$ for the probability for the g-collisions to happen.

Cuckoo Hashing with a Stash and HF's from class $\hat{\mathcal{R}}$

Theorem 3

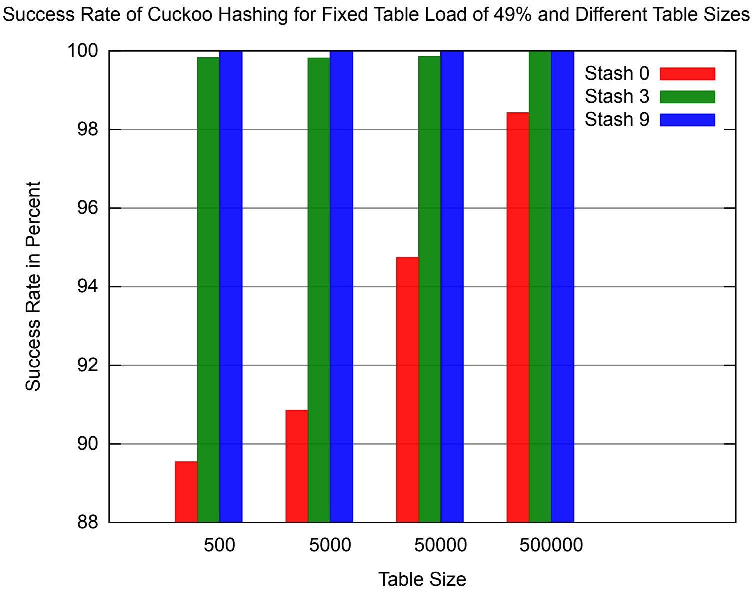
$$\Pr(G(S, h_1, h_2) \text{ is } \ell\text{-bad}) = \mathcal{O}(n \cdot r^{-\ell}).$$

For $r = n^{\beta}$, $\frac{1}{2} < \beta < 1$ and $\ell = 2(s + 2)$.

Corollary

$$\Pr(G(S, h_1, h_2) \text{ is } \ell\text{-bad}) = \mathcal{O}(n^{-(s+1)})$$

Practical Stash Sizes



Conclusion

- stash of size s reduces failure probability drastically ($\mathcal{O}(n^{-1}) \to \mathcal{O}(n^{-(s+1)})$): New proof.
- analysis valid for constant-time, $o(n)$-space class $\hat{\mathcal{R}}$.
- a stash size of only 9 helps us to almost completely avoid rehashes in practical scenarios.

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